

# Stratification and bundle structure of the set of general $(A, B)$ -invariant subspaces

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## Abstract

Given  $(A, B) \in \text{Hom}(\mathbb{C}^{n+m}, \mathbb{C}^n)$ , we prove that the set of  $(A, B)$ -invariant subspaces having a fixed Brunovsky-Kronecker structure is a submanifold of the corresponding grassman manifold, and we compute its dimension. Also, we prove that the set of all  $(A, B)$ -invariant subspaces having a fixed dimension is connected, provided that  $(A, B)$  has only one eigenvalue.

## 1 Introduction

Given linear maps  $A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$  and  $B : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ , a subspace  $W$  of  $\mathbb{C}^n$  is called  $(A, B)$ -invariant if there is a linear map  $R : \mathbb{C}^n \longrightarrow \mathbb{C}^m$  such that  $W$  is  $(A + BR)$ -invariant.  $(A, B)$ -invariant subspaces play a key role in solving many problems in the Linear Systems Theory. See for example [7] and [9], where the algebraic structure of such subspaces is studied and some interesting applications to the Disturbance Decoupling Problem and the Output Stabilization Problem are presented.

One considers the notion of  $\begin{pmatrix} A \\ C \end{pmatrix}$ -invariant subspaces by duality: given  $A$  as above and a linear map  $C : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ , a subspace  $W$  of  $\mathbb{C}^n$  is called  $\begin{pmatrix} A \\ C \end{pmatrix}$ -invariant if  $W^\perp$  is  $(A^*, C^*)$ -invariant. It is easy to see that a coordinate free definition of such kind of subspaces is the following one which is better adapted to our geometric study: given a finite dimensional vector space  $\mathfrak{X}$ , a linear subspace  $\mathcal{Y}$  of  $\mathfrak{X}$  and a linear map  $f : \mathcal{Y} \longrightarrow \mathfrak{X}$ , a subspace  $\mathcal{S}$  of  $\mathcal{Y}$  is called  $f$ -invariant if  $f(\mathcal{S}) \cap \mathcal{Y} \subset \mathcal{S}$ . Of course, if  $\mathcal{Y} = \mathfrak{X}$  this definition is the usual one for endomorphisms. Then, by duality we shall transfer our results to the set of  $(A, B)$ -invariant subspaces.

is only the Jordan part in it, which is equivalent to  $\mathcal{Y}$  being invariant,  $f(\mathcal{Y}) \subset \mathcal{Y}$ , or there is not the Jordan part in it, which is equivalent to  $f$  being observable, this is to say, if  $\begin{pmatrix} A \\ C \end{pmatrix}$  is the matrix of  $f$ ,  $\begin{pmatrix} A \\ C \end{pmatrix}$  is an observable pair.

In the first case we can suppose that  $\mathcal{Y} = \mathfrak{X}$ . This case has been studied by Shayman ([8]). In this paper the topological and differentiable structure of the set of invariant subspaces of an endomorphism is studied. Shayman shows that this set is a connected topological space but it is not, in general, a manifold. However, the subset consisting of the invariant subspaces with a fixed cyclic structure is a regular connected submanifold of the corresponding grassman manifold.

The second case ( $f$  observable) has been studied in [5]. In this paper we have proved that if  $\begin{pmatrix} A \\ C \end{pmatrix}$  is observable, the set of  $\begin{pmatrix} A \\ C \end{pmatrix}$ -invariant subspaces having a fixed Brunovsky structure is a connected manifold and we have computed its dimension.

Now we tackle the case of a general linear map  $f : \mathcal{Y} \longrightarrow \mathfrak{X}$ . For such linear maps, in an analogous way to [5] and [8], one can consider the set of  $d$ -dimensional  $f$ -invariant subspaces  $\mathcal{S}$  of  $\mathcal{Y}$  such that the restriction  $\hat{f} : \mathcal{S} \longrightarrow \mathfrak{X}$  has a given Brunovsky form  $\begin{pmatrix} M \\ F \\ 0 \end{pmatrix}$ , compatible with that of  $f : \mathcal{Y} \longrightarrow \mathfrak{X}$ . Denote this set by  $\text{Inv}(f; (M, F))$ . Our goal is to show that it is a regular submanifold of the grassman manifold  $Gr_d(\mathcal{Y})$  (th. 3.11), to compute its dimension (th. 4.3) and to prove that if  $f$  has only one eigenvalue, it is connected (prop. 4.4), as well as the set  $\text{Inv}_d(f)$  of  $d$ -dimensional  $f$ -invariant subspaces is a connected topological space.

We remark that in order to achieve this, and since for a map defined on a subspace one has not a primary decomposition, it is not possible to adapt the method in [5] where the study is reduced to the case where  $f$  is nilpotent. Here, the differentiable structure of  $\text{Inv}(f; (M, F))$  is obtained by means of a description of it as a homogeneous space (th. 3.9), and the computation of its dimension as well as the proofs concerning connectivity are based on a bundle structure over it (th. 4.1).

The structure of the paper is as follows. Section 2 contains the notation used in the sequel, and the definitions of the sets  $\text{Inv}_d(f)$  and  $\text{Inv}(f; (M, F))$ .

In section 3 we characterize the  $B$ -bases of the subspaces in  $\text{Inv}(f; (M, F))$ , or rather their coordinates in a given  $B$ -basis of  $f$ : if we write them as the columns of a matrix, they are just the solutions of the matrix system (a)-(b)-(c) in th. 3.1.

By means of this characterization, we study the differentiable structure of  $\text{Inv}(f; (M, F))$ : we obtain a description of it as a homogeneous space (th. 3.9), in an analogous way to the grassman manifold  $Gr_d(\mathcal{Y})$ ; then, we use this analogy to prove that  $\text{Inv}(f; (M, F))$  is embedded in  $Gr_d(\mathcal{Y})$  (th. 3.11).

As an application, we study the existence of global Brunovsky bases for a parametrized family of subspaces in  $\text{Inv}(f; (M, F))$ .

In section 4 we compute the dimension of  $\text{Inv}(f; (M, F))$  and we prove that  $\text{Inv}(f; (M, F))$  and  $\text{Inv}_d(f)$  are connected provided that  $f_\infty$  (see section 2 for the definition) has only one eigenvalue.

be transferred by duality to the case of  $(A, B)$ -invariant subspaces.

$\mathcal{M}_{p,q}(\mathbb{C})$  will denote the set of complex matrices having  $p$ -rows and  $q$ -columns, and  $\mathcal{M}_{p,q}^*(\mathbb{C})$  the ones having maximal rank. If  $p = q$ , we will write simply  $\mathcal{M}_p(\mathbb{C})$  and  $\mathcal{M}_p^*(\mathbb{C})$  respectively. The latter, with the group structure of matrix multiplication, is the linear group  $\text{Gl}(\mathbb{C}^p)$ .

For any  $\mathbb{C}$ -vector space  $\mathcal{Z}$ ,  $\text{Gr}_d(\mathcal{Z})$  will denote the grassman manifold of  $d$ -dimensional subspaces of  $\mathcal{Z}$ .

In all the paper,  $B(\dots)$  means Brunovsky- $(\dots)$

## 2 The set $\text{Inv}(\mathbf{f}; (\mathbf{M}, \mathbf{F}))$ of $\mathbf{f}$ -invariant subspaces having the same $\mathbf{B}$ -matrix

We fix an  $(n + m)$ -dimensional vector space  $\mathfrak{X}$  over the complex numbers  $\mathbb{C}$ , an  $n$ -dimensional linear subspace  $\mathcal{Y} \subset \mathfrak{X}$ , and a linear map  $f : \mathcal{Y} \longrightarrow \mathfrak{X}$  defined on it. We recall that a subspace  $\mathcal{S} \subset \mathcal{Y}$  is called  $f$ -invariant if  $f(\mathcal{S}) \cap \mathcal{Y} \subset \mathcal{S}$ .

**Definition 2.1** *Let  $\text{Inv}(f)$  the set of  $f$ -invariant subspaces, and  $\text{Inv}_d(f)$  the subset of those having dimension  $d$ , so that*

$$\text{Inv}(f) = \bigcup_{d \leq n} \text{Inv}_d(f)$$

Our aim is to stratify  $\text{Inv}_d(f)$  according to the  $B$ -matrix of the restrictions  $\hat{f} : \mathcal{S} \longrightarrow \mathfrak{X}$ ,  $\mathcal{S} \in \text{Inv}_d(f)$ . In order to do that, we fix some notation.

We denote by  $\mathcal{Y}_\infty$  the maximal subspace of  $\mathcal{Y}$  such that  $f(\mathcal{Y}_\infty) \subset \mathcal{Y}_\infty$ , or in control systems terminology, the unobservable subsystem of  $f : \mathcal{Y} \longrightarrow \mathfrak{X}$ . In this terminology, if  $\mathcal{Y}_\infty = \{0\}$ ,  $f$  is said to be observable. We write  $k_\infty = \dim \mathcal{Y}_\infty$ , and  $f_\infty : \mathcal{Y}_\infty \longrightarrow \mathcal{Y}_\infty$  the restriction of  $f$  to  $\mathcal{Y}_\infty$ .

We recall (see [7] or [3]) that there exist so-called  $B$ -bases of  $f : \mathcal{Y} \longrightarrow \mathfrak{X}$  of the form  $B = (B_0, B_\infty, B_E, B_A)$ , where  $B$  is a basis of  $\mathfrak{X}$  and  $(B_0, B_\infty)$  is a basis of  $\mathcal{Y}$  such that:  $B_\infty$  is a Jordan basis of  $f_\infty$ ;  $B_0$  is formed by so-called  $B$ -chains  $w_i, f(w_i), \dots, f^{k_i-1}(w_i)$ ,  $1 \leq i \leq r$ ,  $k_1 \geq \dots \geq k_r$ ;  $B_E$  is the family  $f^{k_1}(w_1), \dots, f^{k_r}(w_r)$ , formed by the ends of the  $B$ -chains;  $B_A$  is arbitrary.

The integers  $k_1, \dots, k_r$  do not depend on the choice of the  $B$ -basis, and are called the  $B$ -indices of  $f$ . We write  $k_0 = k_1 + \dots + k_r = n - k_\infty$ .

Then, in any bases  $(B_0, B_\infty)$  and  $B$  of this kind, the matrix of  $f$  is  $\begin{pmatrix} N \\ E \\ 0 \end{pmatrix} \in \mathcal{M}_{n+m,n}(\mathbb{C})$ , where:  $N = \text{diag} \{N_0, N_\infty\}$ ,  $N_0 = \text{diag} \{N_1, \dots, N_r\}$ , each  $N_i$  being the standard nilpotent  $k_i$ -square matrix,  $N_i = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$  and  $N_\infty \in \mathcal{M}_{k_\infty}(\mathbb{C})$  the Jordan matrix of  $f_\infty$ ; and  $E = (E_0, 0)$ ,  $E_0 = \text{diag} \{E_1, \dots, E_r\}$ , each  $E_i = (0 \ 0 \ \dots \ 0 \ 1)$  being a  $k_i$ -row matrix. This is called the  $B$ -matrix of  $f$ .

vectors of  $\mathfrak{X}$  (respectively  $\mathcal{Y}$ ) with the  $(n+m)$ -columns (respectively,  $n$ -columns) matrices of their coordinates in this basis. In the same sense, we identify  $\text{Gr}_d(\mathcal{Y})$  with  $\text{Gr}_d(\mathbf{C}^n)$ , for any  $d$ .

Finally, we consider  $B$ -matrices  $\begin{pmatrix} M \\ F \\ 0 \end{pmatrix} \in \mathcal{M}_{n+m,d}(\mathbf{C})$ , where, in an analogous way to  $\begin{pmatrix} N \\ E \\ 0 \end{pmatrix}$ ,  $M = \text{diag} \{M_0, M_\infty\}$ ,  $M_0 = \text{diag} \{M_1, \dots, M_s\}$ , each  $M_i$  being the standard nilpotent  $h_i$ -square matrix, etc. For each of these matrices, we have:

**Definition 2.2** *With the above notation, we denote by  $\text{Inv}_d(f; (M, F))$ , or simply  $\text{Inv}(f; (M, F))$  if no confusion is possible, the set of subspaces  $\mathcal{S} \in \text{Inv}_d(f)$  such that the  $B$ -matrix of the restriction  $\hat{f} : \mathcal{S} \rightarrow \mathfrak{X}$  is  $\begin{pmatrix} M \\ F \\ 0 \end{pmatrix}$ . Hence*

$$\text{Inv}_d(f) = \bigcup_{M, F} \text{Inv}(f; (M, F)).$$

**Remark 2.3** We assume that the following conditions that guarantee that  $\text{Inv}(f; (M, F)) \neq \emptyset$  are verified (see [8], [1], [2]):

- (a)  $s \leq r$ , and  $h_i \leq k_i$  for  $i = 1, 2, \dots$
- (b) the eigenvalues of  $M_\infty$  are also eigenvalues of  $N_\infty$ , and for each one the corresponding Segre characteristics  $(\eta_1(\lambda), \eta_2(\lambda), \dots)$  and  $(\varepsilon_1(\lambda), \varepsilon_2(\lambda), \dots)$  verify:  $\eta_i(\lambda) \leq \varepsilon_i(\lambda)$ , for  $i = 1, 2, \dots$

### 3 The stratification of $\text{Inv}_d(f)$

In this section we are going to see that the partition in definition 2.2 is in fact a stratification. This is to say, that each stratum  $\text{Inv}_d(f; (M, F))$  is a submanifold of  $\text{Gr}_d(\mathcal{Y})$ . Its differentiable structure is the natural generalization of the one in [5], where  $f$  is assumed observable. We recall that construction; the proofs are direct generalizations of the ones in [5].

Let  $\Phi$  be the map

$$\Phi : \mathcal{M}_{n,d}^*(\mathbf{C}) \rightarrow \text{Gr}_d(\mathcal{Y})$$

defined as follows: if  $X \in \mathcal{M}_{n,d}^*(\mathbf{C})$ ,  $\Phi(X)$  is the subspace  $\mathcal{S} \in \text{Gr}_d(\mathcal{Y})$  spanned by the columns of  $X$ . For simplicity, we say that  $X$  is a basis of  $\mathcal{S} = \Phi(X)$ . In addition, we say that  $X$  is a  $B$ -basis of  $\mathcal{S} = \Phi(X)$  if it can be extended to a  $B$ -basis of  $\hat{f} : \mathcal{S} \rightarrow \mathfrak{X}$ .

**Theorem 3.1** *With the above notation:*

1. Let  $\mathcal{S} \in \text{Gr}_d(\mathcal{Y})$  such that  $\begin{pmatrix} M \\ F \\ 0 \end{pmatrix}$  is the  $B$ -matrix of the restriction  $\hat{f} : \mathcal{S} \rightarrow \mathfrak{X}$ . If  $X$  is a  $B$ -matrix of  $\mathcal{S}$ , then

(b)  $EX = EXF^tF$

Moreover,  $\mathcal{S}$  is  $f$ -invariant if and only if:

(c)  $EXF^t$  has maximal rank.

2. Conversely, let  $X \in \mathcal{M}_{n,d}^*(\mathbf{C})$ . If  $X$  verifies the conditions (a), (b) and (c) above, then  $\mathcal{S} = \Phi(X) \in \text{Inv}(f; (M, F))$ .

This result motivates the following:

**Definition 3.2** We denote by  $\mathcal{M}((N, E); (M, F))$ , or simply by  $\mathcal{M}$  if no confusion is possible, the set of matrices  $X \in \mathcal{M}_{n,d}^*(\mathbf{C})$  which verify conditions (a), (b) and (c) in theorem 3.1.

Obviously,  $\mathcal{M}$  is a submanifold of  $\mathcal{M}_{n,d}^*$ . In fact, it is an open subset of a linear subvariety of  $\mathcal{M}_{n,d}$ .

With this notation, from theorem 3.1 we have

**Corollary 3.3** With the above notation, we have:

$$\Phi(\mathcal{M}((N, E); (M, F))) = \text{Inv}(f; (M, F)).$$

In general,  $\Phi$  is not injective. In fact, we have that  $\Phi(X) = \Phi(X')$  if and only if  $X' = XT$  for some  $T \in \text{Gl}(\mathbf{C}^d)$ . If only matrices in  $\mathcal{M}$  are considered, we have:

**Proposition 3.4** Let  $X, X' \in \mathcal{M}$ . Then,  $\Phi(X) = \Phi(X')$  if and only if there is  $T \in \text{Gl}(\mathbf{C}^d)$  such that:  $X' = XT$ , and

$$(a') \quad MT = TM + MTF^tF$$

$$(b') \quad FT = FTF^tF$$

This proposition suggests the following definition:

**Definition 3.5** We denote by  $\mathcal{G}(M, F)$ , or simply by  $\mathcal{G}$  if no confusion is possible, the set of matrices  $T \in \text{Gl}(\mathbf{C}^d)$  which verify conditions (a') and (b') in proposition 3.4.

**Lemma 3.6** With the above notation, if  $T \in \mathcal{G}$ , then

$$(c') \quad FTF^t \text{ has maximal rank.}$$

In fact,  $(FTF^t)^{-1} = FT^{-1}F^t$ .

**Remark 3.7** Because of the last lemma, we can identify  $\mathcal{G}$  with  $\mathcal{M}((M, F); (M, F))$ . However, we are mainly interested in the group structure of  $\mathcal{G}$ .

1.  $\mathcal{G}$  is a subgroup of  $\text{Gl}(\mathbf{C}^d)$
2.  $\mathcal{G}$  acts freely on  $\mathcal{M}$  on the right by matrix multiplication.

Since  $\mathcal{G}$  acts on  $\mathcal{M}$  we can consider the orbit  $X\mathcal{G}$  of an element  $X \in \mathcal{M}$ , which is the set  $\{XT; T \in \mathcal{G}\}$ . Now a differentiable structure in  $\text{Inv}(f; (M, F))$  can be defined by means of the following theorem:

**Theorem 3.9** *Let  $\mathcal{M}/\mathcal{G}$  be the set of orbits under the action 3.8, and  $\tilde{\Phi}$  the map induced on it by  $\Phi$ . Then:*

1.  $\tilde{\Phi} : \mathcal{M}/\mathcal{G} \longrightarrow \text{Inv}(f; (M, F))$  is a bijection.
2. The orbit space  $\mathcal{M}/\mathcal{G}$  has a differentiable structure such that the natural projection  $\pi : \mathcal{M} \longrightarrow \mathcal{M}/\mathcal{G}$  is a submersion.

**Remark 3.10** *In the conditions of the above theorem, it is known that the following properties are verified:*

1. Each orbit  $X\mathcal{G} = \{XT, T \in \mathcal{G}\}$  is a closed submanifold of  $\mathcal{M}$ , diffeomorphic to  $\mathcal{G}$ .
2. For any differentiable manifold  $\mathcal{N}$ , a map  $\psi : \mathcal{M}/\mathcal{G} \longrightarrow \mathcal{N}$  is smooth if and only if  $\psi \circ \pi$  is smooth. In particular,  $\tilde{\Phi} : \mathcal{M}/\mathcal{G} \longrightarrow \text{Gr}_d(\mathcal{Y})$  is smooth.
3. The submersion  $\pi : \mathcal{M} \longrightarrow \mathcal{M}/\mathcal{G}$  is a principal bundle with structural group  $\mathcal{G}$ .
4.  $\dim(\mathcal{M}/\mathcal{G}) = \dim \mathcal{M} - \dim \mathcal{G}$ .

Finally, we are going to see that the inclusion  $\text{Inv}_d(f; (M, F)) \subset \text{Gr}_d(\mathcal{Y})$  is in fact an embedding (it was not tackled in [5]).

**Theorem 3.11** *With the differentiable structure defined by means of 3.9,  $\text{Inv}_d(f; (M, F))$  is a submanifold of  $\text{Gr}_d(\mathcal{Y})$ .*

*Proof.* Because of the identifications above, this is equivalent to proving that  $\mathcal{M}/\mathcal{G}$  is a submanifold of  $\text{Gr}_d(\mathbf{C}^n) = \mathcal{M}_{n,d}^*(\mathbf{C})/\text{Gl}(\mathbf{C}^d)$ . For simplicity, we shall write  $\mathcal{M}^* \equiv \mathcal{M}_{n,d}^*(\mathbf{C})$ ,  $G \equiv \text{Gl}(\mathbf{C}^d)$ . We consider the commutative diagram

$$\begin{array}{ccc}
 \overline{\mathcal{M}} & \xrightarrow{\tilde{j}} & \mathcal{M}_{n,d}(\mathbf{C}) \\
 \cup & & \cup \\
 \mathcal{M} & \xrightarrow{j} & \mathcal{M}^* \\
 \pi \downarrow & & \downarrow \pi' \\
 \mathcal{M}/\mathcal{G} & \xrightarrow{\tilde{j}} & \mathcal{M}^*/G = \text{Gr}_d(\mathbf{C}^n)
 \end{array}$$

defined by (a,b) of 3.1,  $\tilde{j}$  and  $j$  are the natural embeddings, and  $j$  is the injection (see (1) of 3.9) induced by  $j$ .

It is sufficient to obtain a local representation  $\hat{j} = \sigma' \circ \tilde{j} \circ \sigma^{-1} : \mathcal{U} \rightarrow V$  of  $\tilde{j}$ , where  $\sigma : \tilde{\mathcal{U}} \rightarrow \mathcal{U}$  and  $\sigma' : \tilde{V} \rightarrow V$  are local charts of  $\mathcal{M}/\mathcal{G}$  and  $\mathcal{M}^*/G$  respectively, such that  $\mathcal{U}$  is a subspace of  $V$ , and  $\hat{j}$  is the natural inclusion. In fact, given  $X \in \mathcal{M}$ , we shall construct local charts at  $\pi(X)$  and  $\pi'(X)$  of this kind as local sections of the submersions  $\pi$  and  $\pi'$ .

In order to that, let  $L$  be the linear variety of  $\overline{\mathcal{M}}$  orthogonal at  $X$  to its  $\mathcal{G}$ -orbit  $X\mathcal{G} = \pi^{-1}(\pi(X))$  (with regard to the usual hermitic product in  $\mathcal{M}_{n,d}(\mathbf{C})$ ). Then, there are open neighbourhoods  $\mathcal{U}$  of  $X$  in  $L$  and  $\tilde{\mathcal{U}}$  of  $\pi(X)$  in  $\mathcal{M}/\mathcal{G}$  such that the inverse of the diffeomorphism  $\pi : \mathcal{U} \rightarrow \tilde{\mathcal{U}}$  is a local chart  $\sigma$  for  $\pi(X)$ .

Next, if  $XG = \pi'^{-1}(\pi'(X))$  is the  $G$ -orbit of  $X$ , it is clear (prop. 3.4) that  $L \cap (XG) = \{X\}$ , so that there is a linear variety  $W$  of  $\mathcal{M}_{n,d}(\mathbf{C})$ , minitransversal to  $XG$  and containing  $L$ . Then, there are open neighbourhoods  $V$  of  $X$  in  $W$  and  $\tilde{V}$  of  $\pi'(X)$  in  $\mathcal{M}^*/G$  such that the inverse of the diffeomorphism  $\pi' : V \rightarrow \tilde{V}$  is a local chart  $\sigma'$  for  $\pi'(X)$ .

Finally, shrinking the neighbourhoods if necessary, it is clear that the embedding  $\hat{j} = \sigma' \circ \tilde{j} \circ \sigma^{-1} : \mathcal{U} \rightarrow V$  represents the injection  $\tilde{j} : \tilde{\mathcal{U}} \rightarrow \tilde{V}$ , q.e.d. ■

**Corollary 3.12** *The partition*

$$\text{Inv}_d(f) = \bigcup_{M,F} \text{Inv}_d(f; (M, F)) \subset \text{Gr}_d(\mathcal{Y})$$

*is a stratification.*

As a first application, we study the existence of global differentiable  $B$ -bases for a differentiable parametrized family of subspaces in  $\text{Inv}(f; (M, F))$ .

We consider a manifold  $\mathcal{W}$ , and a family  $\mathcal{S}(t)$ ,  $t \in \mathcal{W}$ , of subspaces in  $\text{Inv}(f; (M, F))$ , differentiable parametrized over  $\mathcal{W}$ . It can simply be represented as a differentiable map  $\mathcal{S}(t) : \mathcal{W} \rightarrow \text{Inv}(f; (M, F))$ . We know that, for each  $t \in \mathcal{W}$ , there are some  $B$ -bases  $X_t \in \mathcal{M}((N, E); (M, F))$  of  $\mathcal{S}(t)$ . We show that, locally, it is possible to choose one of these  $B$ -bases  $X(t)$  for each  $t$ , in such a way that it depends differentiably of  $t$ . Moreover, this result globalizes to all  $\mathcal{W}$  if it is contractible. The proof is a direct generalization of the one of 5.1 in [5].

**Proposition 3.13** *Let  $\mathcal{W}$  be a manifold, and  $\mathcal{S}(t) : \mathcal{W} \rightarrow \text{Inv}(f; (M, F))$  a differentiable map.*

1. *For each  $t_0 \in \mathcal{W}$  there exist an open neighbourhood  $\mathcal{U}$  of  $t_0$  in  $\mathcal{W}$ , and a differentiable map  $X(t) : \mathcal{U} \rightarrow \mathcal{M}((N, E); (M, F))$  such that  $X(t)$  is a  $B$ -basis of  $\mathcal{S}(t)$ , for all  $t \in \mathcal{U}$ .*
2. *If in addition  $\mathcal{W}$  is contractible, the statement in part 1 holds for  $\mathcal{U} = \mathcal{W}$ .*

Further properties can be derived from the following bundle structure, which reduces the problem to the simplest cases of  $f$  being an endomorphism (see [8]) and  $f$  being observable (see [5]).

In order to do that, let consider the maps

$$\begin{aligned} f_\infty : \mathcal{Y}_\infty &\longrightarrow \mathcal{Y}_\infty \\ \tilde{f} : \mathcal{Y}/\mathcal{Y}_\infty &\longrightarrow \mathfrak{X}/\mathcal{Y}_\infty \end{aligned}$$

induced by  $f$  in a natural way. Clearly  $f_\infty$  is an endomorphism and  $\tilde{f}$  is an observable map, so that  $\text{Inv}(f_\infty; M_\infty)$  has been studied in [8] and  $\text{Inv}(\tilde{f}; (M_0, F_0))$ , in [5]. Notice that their reduced matrices are  $N_\infty$  and  $\begin{pmatrix} N_0 \\ E_0 \\ 0 \end{pmatrix}$ , respectively.

The key point is the following bundle structure:

**Theorem 4.1** *The map*

$$\begin{aligned} \theta : \text{Inv}(f; (M_0, F_0)) \times \text{Inv}(f; M_\infty) &\longrightarrow \text{Inv}(f; (M, F)) \\ \theta(\mathcal{S}_0, \mathcal{S}_\infty) &= \mathcal{S}_0 \oplus \mathcal{S}_\infty \end{aligned}$$

*is a  $(h_\infty s)$ -dimensional vector bundle.*

*Proof.* Notice that if  $\mathcal{S}_0 \in \text{Inv}(f; (M_0, F_0))$  and  $\mathcal{S}_\infty \in \text{Inv}(f_\infty; M_\infty)$ , then  $\mathcal{S}_0 \cap \mathcal{Y}_\infty = \{0\}$ ,  $\mathcal{S}_\infty \subset \mathcal{Y}_\infty$  and  $\mathcal{S}_0 \oplus \mathcal{S}_\infty \in \text{Inv}(f; (M, F))$ , so that  $\theta$  is well defined.

We will see that for any  $\mathcal{S} \in \text{Inv}(f; (M, F))$  there is an open neighbourhood  $\mathcal{U}$  of  $\mathcal{S}$ , and a diffeomorphism

$$\varphi : \mathcal{U} \times \mathcal{M}_{h_\infty, s} \longrightarrow \theta^{-1}(\mathcal{U}) \subset \text{Inv}(f; (M_0, F_0)) \times \text{Inv}(f; M_\infty)$$

such that  $\theta \circ \varphi : \mathcal{U} \times \mathcal{M}_{h_\infty, s} \longrightarrow \mathcal{U}$  is the natural projection.

Let  $\sigma : \mathcal{U} \longrightarrow \mathcal{M}$  be a smooth local section of the submersion  $\mathcal{M} \longrightarrow \mathcal{M}/\mathcal{G} \cong \text{Inv}(f; (M, F))$ . Thus, for each  $\mathcal{T} \in \mathcal{U}$ ,  $\sigma(\mathcal{T})$  is a  $B$ -basis of  $\mathcal{T}$ , of the form  $\sigma(\mathcal{T}) = (\hat{B}_0(\mathcal{T}), \hat{B}_\infty(\mathcal{T}))$  as in the proof of 3.1, so that  $[\hat{B}_0(\mathcal{T})] \in \text{Inv}(f; (M_0, F_0))$  and  $[\hat{B}_\infty(\mathcal{T})] \in \text{Inv}(f; M_\infty)$ , where  $[(\dots)]$  means the subspace spanned by  $(\dots)$ . We denote by  $u_1(\mathcal{T}), \dots, u_s(\mathcal{T})$  the generators of the  $B$ -chains in  $\hat{B}_0(\mathcal{T})$ , and by  $e_1(\mathcal{T}), \dots, e_{h_\infty}(\mathcal{T})$  the vectors of the basis  $\hat{B}_\infty(\mathcal{T})$ .

Then, we define  $\varphi$  as follows. Given  $\mathcal{T} \in \mathcal{U}$  and  $Z = (z_j^i) \in \mathcal{M}_{h_\infty, s}$ , let  $v_j(\mathcal{T}) = u_j(\mathcal{T}) + \sum_{1 \leq i \leq h_\infty} z_j^i e_i(\mathcal{T})$ ,  $1 \leq j \leq s$ . It is trivial, that  $v_1(\mathcal{T}), \dots, v_s(\mathcal{T})$  generate linearly independent  $B$ -chains having the same length as those generated by  $u_1(\mathcal{T}), \dots, u_s(\mathcal{T})$  respectively; we denote them by  $\hat{B}_0(\mathcal{T}) + Z\hat{B}_\infty(\mathcal{T})$ , that is to say:  $\hat{B}_0(\mathcal{T}) + Z\hat{B}_\infty(\mathcal{T}) = \{v_j(\mathcal{T}), \hat{f}(v_j(\mathcal{T})), \dots, \hat{f}^{h_j-1}(v_j(\mathcal{T})); 1 \leq j \leq s\}$ . Hence,  $[\hat{B}_0(\mathcal{T}) + Z\hat{B}_\infty(\mathcal{T})] \in \text{Inv}(f; (M_0, F_0))$ . Then, we define

$$\varphi(\mathcal{T}, Z) = ([\hat{B}_0(\mathcal{T}) + Z\hat{B}_\infty(\mathcal{T})], [\hat{B}_\infty(\mathcal{T})])$$



$$(\theta \circ \varphi)(\mathcal{T}, Z) = [\hat{B}_0(\mathcal{T}) + Z\hat{B}_\infty(\mathcal{T})] \oplus [\hat{B}_\infty(\mathcal{T})] = [\hat{B}_0(\mathcal{T})] \oplus [\hat{B}_\infty(\mathcal{T})] = \mathcal{T}$$

Also, it is easy to see that  $\varphi$  is injective.

To show that  $\varphi$  is surjective and that  $\varphi^{-1}$  is smooth, we shall construct a local inverse. That is to say, given  $(\mathcal{T}_0, \mathcal{T}_\infty) \in \theta^{-1}(\mathcal{U})$ , we shall obtain a neighbourhood  $V \subset \theta^{-1}(\mathcal{U})$  and a smooth map  $\eta : V \longrightarrow \mathcal{U} \times \mathcal{M}_{h_\infty, s}$  such that  $\varphi \circ \eta$  be the identity.

In order to do that, let  $\sigma_0 : \mathcal{U}_0 \longrightarrow \mathcal{M}((N, E); (M_0, F_0))$  and  $\sigma_\infty : \mathcal{U}_\infty \longrightarrow \mathcal{M}((N, E); M_\infty)$  be local sections at  $\mathcal{T}_0$  and  $\mathcal{T}_\infty$  of the respective submersions, so that for each  $\mathcal{T}'_0 \in \mathcal{U}_0$  and for each  $\mathcal{T}'_\infty \in \mathcal{U}_\infty$ , the images  $\sigma_0(\mathcal{T}'_0)$  and  $\sigma_\infty(\mathcal{T}'_\infty)$  are  $B$ -bases of  $\mathcal{T}'_0$  and  $\mathcal{T}'_\infty$  respectively.

We take  $V = (\mathcal{U}_0 \times \mathcal{U}_\infty) \cap \theta^{-1}(\mathcal{U})$ , and we define the first component  $\eta_1 : V \longrightarrow \mathcal{U}$  of  $\eta$  by means of:  $\eta_1(\mathcal{T}'_0, \mathcal{T}'_\infty) = \theta(\mathcal{T}'_0, \mathcal{T}'_\infty) = \mathcal{T}'_0 \oplus \mathcal{T}'_\infty$ . Notice that we have two  $B$ -bases of  $\mathcal{T}'_0 \oplus \mathcal{T}'_\infty : (\sigma_0(\mathcal{T}'_0), \sigma_\infty(\mathcal{T}'_\infty))$ , and  $\sigma(\mathcal{T}'_0 \oplus \mathcal{T}'_\infty) = (\hat{B}_0(\mathcal{T}'_0 \oplus \mathcal{T}'_\infty), \hat{B}_\infty(\mathcal{T}'_0 \oplus \mathcal{T}'_\infty))$ . Clearly,  $[\hat{B}_\infty(\mathcal{T}'_0 \oplus \mathcal{T}'_\infty)] = [\sigma_\infty(\mathcal{T}'_\infty)] = \mathcal{T}'_\infty$ . But, in general,  $[\hat{B}_0(\mathcal{T}'_0 \oplus \mathcal{T}'_\infty)] \neq [\sigma_0(\mathcal{T}'_0)] = \mathcal{T}'_0$ . We will define  $\eta_2(\mathcal{T}'_0, \mathcal{T}'_\infty) = Z \in \mathcal{M}_{h_\infty, s}$  in such a way that:  $[\hat{B}_0(\mathcal{T}'_0 \oplus \mathcal{T}'_\infty) + Z\hat{B}_\infty(\mathcal{T}'_0 \oplus \mathcal{T}'_\infty)] = \mathcal{T}'_0$ . Then, the proof will be completed because:  $(\varphi \circ \eta)(\mathcal{T}'_0, \mathcal{T}'_\infty) = \varphi(\mathcal{T}'_0 \oplus \mathcal{T}'_\infty, Z) = ([\hat{B}_0(\mathcal{T}'_0 \oplus \mathcal{T}'_\infty) + Z\hat{B}_\infty(\mathcal{T}'_0 \oplus \mathcal{T}'_\infty)], [\hat{B}_\infty(\mathcal{T}'_0 \oplus \mathcal{T}'_\infty)]) = (\mathcal{T}'_0, \mathcal{T}'_\infty)$ .

To obtain this  $Z$ , if  $u_1, \dots, u_s$  are now the generators of the  $B$ -chains in  $\hat{B}_0(\mathcal{T}'_0 \oplus \mathcal{T}'_\infty)$ , we will determine  $z_1, \dots, z_s \in \mathcal{T}'_\infty$  such that the subspace spanned by the  $B$ -chains generated by  $u_1 + z_1, \dots, u_s + z_s$  be just  $\mathcal{T}'_0$ . In order to do that, we shall consider separately the  $B$ -chains of different length. Let  $v_1, \dots, v_s$  be the generators of the  $B$ -chains in  $\sigma_0(\mathcal{T}'_0)$ , and  $(s_1, \dots, s_{k_1})$  the partition of  $s$  according to the lengths  $k_1, k_1 - 1, \dots, 1$  of the corresponding  $B$ -chain, that is to say:  $v_1, \dots, v_{s_1}$  (and hence  $u_1, \dots, u_{s_1}$ ) generate  $B$ -chains having length  $k_1$ ;  $v_{s_1+1}, \dots, v_{s_2}$  (and hence  $u_{s_1+1}, \dots, u_{s_2}$ ) generate  $B$ -chains having length  $k_1 - 1$ ; etc. If, on the other hand, we write  $\mathcal{T}'_i = (\mathcal{T}'_0 \oplus \mathcal{T}'_\infty) \cap f^{-i}(\mathcal{T}'_0 \oplus \mathcal{T}'_\infty)$ ,  $1 \leq i \leq k_1$ , we have:

$$\begin{aligned} \mathcal{T}'_{k_1} &= \mathcal{T}'_\infty \\ \mathcal{T}'_{k_1-1} &= \mathcal{T}'_\infty \oplus [v_1, \dots, v_{s_1}] \\ \mathcal{T}'_{k_1-2} &= \mathcal{T}'_\infty \oplus [v_1, \dots, v_{s_1}] \oplus [f(v_1), \dots, f(v_{s_1})] \oplus \\ &\quad \oplus [v_{s_1+1}, \dots, v_{s_2}] \\ &\text{etc.} \end{aligned}$$

and analogously for  $u_1, \dots, u_s$ . Therefore, there are unique  $z_i \in \mathcal{T}'_\infty$ ,  $1 \leq i \leq s$ , such that:

$$\begin{aligned} u_i + z_i &\in [v_1, \dots, v_{s_1}], \quad 1 \leq i \leq s_1 \\ u_i + z_i &\in [v_1, \dots, v_{s_1}] \oplus [f(v_1), \dots, f(v_{s_1})] \oplus \\ &\quad \oplus [v_{s_1+1}, \dots, v_{s_2}], \quad s_1 + 1 \leq i \leq s_2 \\ &\text{etc.} \end{aligned}$$

Then, we define  $\eta_2(\mathcal{T}'_0, \mathcal{T}'_\infty) = Z \in \mathcal{M}_{h_\infty, s}$  by means of: the columns of  $Z$  are the coordinates of  $z_1, \dots, z_s$  in the basis  $\hat{B}_\infty(\mathcal{T}'_0 \oplus \mathcal{T}'_\infty)$ .

$$\mathcal{T}'_0 = \left( \left[ \widehat{B}_0(\mathcal{T}'_0 \oplus \mathcal{T}'_\infty) + Z\widehat{B}_\infty(\mathcal{T}'_0 \oplus \mathcal{T}'_\infty) \right] \right).$$

We recall that  $\mathcal{T}'_0$  is spanned by the  $B$ -chains generated by  $v_1, \dots, v_s$ , and that  $[\widehat{B}_0(\mathcal{T}'_0 \oplus \mathcal{T}'_\infty) + Z\widehat{B}_\infty(\mathcal{T}'_0 \oplus \mathcal{T}'_\infty)]$  is spanned by the  $B$ -chains generated by  $u_1 + z_1, \dots, u_s + z_s$ . Finally, by the construction of  $z_1, \dots, z_s$ , we have:

$$\begin{aligned} [v_1, \dots, v_{s_1}] &= [u_1 + z_1, \dots, u_{s_1} + z_{s_1}] \\ [v_1, \dots, v_{s_1}] \oplus [f(v_1), \dots, f(v_{s_1})] \oplus [v_{s_1+1}, \dots, v_{s_2}] &= \\ &= [u_1 + z_1, \dots, u_{s_1} + z_{s_1}] \oplus [f(u_1 + z_1), \dots, f(u_{s_1} + z_{s_1})] \oplus \\ &\oplus [u_{s_1+1} + z_{s_1+1}, \dots, u_{s_2} + z_{s_2}] \\ &\text{etc.} \end{aligned}$$

■

The reduction to the known simplest cases  $\text{Inv}(f_\infty; M_\infty)$  and  $\text{Inv}(\tilde{f}; (M_0, F_0))$  is completed by means of the following proposition:

**Proposition 4.2** 1.  $\text{Inv}(f; M_\infty) = \text{Inv}(f_\infty; M_\infty)$ .

2. The map

$$\begin{aligned} \psi : \text{Inv}(f; (M_0, F_0)) &\longrightarrow \text{Inv}(\tilde{f}; (M_0, F_0)) \\ \psi(\mathcal{S}) &= (\mathcal{S} \oplus \mathcal{Y}_\infty) / \mathcal{Y}_\infty \end{aligned}$$

is a  $(k_\infty s)$ -dimensional vector bundle.

*Proof.*

1. It is trivial.

2. Let consider  $\text{Inv}(f; (\overline{M}, \overline{F}))$ , where  $M = \text{diag}\{M_0, N_\infty\}$ ,  $\overline{F} = \text{diag}\{F_0, 0\}$ .

Taking into account that  $\text{Inv}(f; N_\infty) = \{\mathcal{Y}_\infty\}$ , we can identify  $\psi$  with the composition  $\overline{\psi} \circ \overline{\theta}$  where

$$\begin{aligned} \overline{\theta} : \text{Inv}(f; (M_0, F_0)) \times \text{Inv}(f; N_\infty) &\longrightarrow \text{Inv}(f; (\overline{M}, \overline{F})) \\ \overline{\theta}(\mathcal{S}, \mathcal{Y}_\infty) &= \mathcal{S} \oplus \mathcal{Y}_\infty \\ \overline{\psi} : \text{Inv}(f; (\overline{M}, \overline{F})) &\longrightarrow \text{Inv}(\tilde{f}; (M_0, F_0)) \\ \overline{\psi}(\overline{\mathcal{S}}) &= \overline{\mathcal{S}} / \mathcal{Y}_\infty \end{aligned}$$

According to th. 4.1, the map  $\overline{\theta}$  is a  $(k_\infty s)$ -dimensional vector bundle. Let see that  $\overline{\psi}$  is a diffeomorphism. It is straightforward that if  $\overline{\mathcal{S}} \in \text{Inv}(f; (\overline{M}, \overline{F}))$ , then  $\overline{\mathcal{S}} \supset \mathcal{Y}_\infty$  and  $\overline{\mathcal{S}} / \mathcal{Y}_\infty \in \text{Inv}(\tilde{f}; (M_0, F_0))$ , and conversely. Therefore,  $\overline{\psi}$  is the restriction of the diffeomorphism

$$\Psi : \{\overline{\mathcal{S}} \in \text{Gr}_{d+k_\infty}(\mathcal{Y}) : \overline{\mathcal{S}} \supset \mathcal{Y}_\infty\} \longrightarrow \text{Gr}_d(\mathcal{Y} / \mathcal{Y}_\infty), \quad \Psi(\overline{\mathcal{S}}) = \overline{\mathcal{S}} / \mathcal{Y}_\infty.$$

■

**Theorem 4.3** *Let  $f : \mathcal{Y} \longrightarrow \mathfrak{X}$ ,  $M$  and  $F$  be as in section 2. With the above notation:*

$$\begin{aligned} \dim \operatorname{Inv}(f; (M, F)) &= \dim \operatorname{Inv}(\tilde{f}; (M_0, F_0)) + \dim \operatorname{Inv}(f_\infty; M_\infty) + (k_\infty - h_\infty)s = \\ &= \sum_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} \sup\{k_i - h_j + 1, 0\} - \sum_{1 \leq i, j \leq s} \sup\{h_i - h_j + 1, 0\} + \\ &\quad + \dim \operatorname{Inv}(f_\infty; M_\infty) + (k_\infty - h_\infty)s \end{aligned}$$

*Proof.* To obtain the first equality it is sufficient to apply successively 4.1 and 4.2:

$$\begin{aligned} \dim \operatorname{Inv}(f; (M, F)) &= \\ &= \dim \operatorname{Inv}(f; (M_0, F_0)) + \dim \operatorname{Inv}(f; M_\infty) - h_\infty s = \\ &= \dim \operatorname{Inv}(\tilde{f}; (M_0, F_0)) + k_\infty s + \\ &\quad + \dim \operatorname{Inv}(f_\infty; M_\infty) - h_\infty s \end{aligned}$$

Then, the second one follows from 5.3 in [5]. ■

For a second application we recall that in [5] and [8] one proves, respectively, the connectivity of  $\operatorname{Inv}(\tilde{f}; (M_0, F_0))$  and of  $\operatorname{Inv}(f_\infty; M_\infty)$  when the endomorphism  $f_\infty$  has only one eigenvalue. Thus, in a similar way to 4.3, one proves:

**Proposition 4.4** *Any stratum  $\operatorname{Inv}(f; (M, F))$  is connected, provided that the endomorphism  $f_\infty$  has only one eigenvalue.*

Moreover, in [8] one proves that  $\operatorname{Inv}_d(f)$  is connected when  $f$  is an endomorphism having only one eigenvalue. We will generalize this result. Firstly, we consider the case when  $f$  is observable.

**Proposition 4.5** *Let  $f$  be observable. Then  $\operatorname{Inv}_d(f)$  is connected.*

*Proof.* Let  $f$  be observable, and  $k_1 \geq \dots \geq k_r$  its  $B$ -indices. Then, for any  $f$ -invariant subspace  $\mathcal{S} \subset \mathcal{Y}$ , the restriction  $\hat{f} : \mathcal{S} \longrightarrow \mathfrak{X}$  is also observable, so that its  $B$ -matrix is determined by its  $B$ -indices  $h_1 \geq \dots \geq h_s$  (by convenience, we consider  $h_{s+1} = \dots = h_r = 0$ ). Hence, we can note  $\operatorname{Inv}_d(f; (M, F)) \equiv \operatorname{Inv}_s(f; h)$ , and

$$\operatorname{Inv}_d(f) = \cup_h \operatorname{Inv}_d(f; h)$$

where  $h = (h_1, \dots, h_r)$  runs over all the so-called “partitions of  $d$  compatibles with  $k = (k_1, \dots, k_r)$ ”, that is to say:  $h_1 + \dots + h_r = d$ ,  $h_1 \geq \dots \geq h_r \geq 0$ , and  $h_i \leq k_i$  for all  $1 \leq i \leq r$ .

We know that each stratum  $\operatorname{Inv}_d(f; h)$  is connected (prop. 4.4). In order to prove that the union of them is also connected, let consider the stratum  $\operatorname{Inv}_d(f; h_*)$  where  $h_*$  is the only partition such that:

$$h_i \geq h_1 - 1, \quad \text{if } h_i < k_i$$

such that

$$\text{Inv}_d(f; h) \cup \text{Inv}_d(f; h') \cup \text{Inv}_d(f; h'') \cup \dots \cup \text{Inv}_d(f; h_*)$$

is connected.

For that, given  $h \neq h_*$  let be  $\alpha < \beta$  such that

$$\begin{aligned} h_1 &= h_2 = \dots = h_\alpha > h_{\alpha+1} \\ \beta &= \inf\{i : h_i < h_1 - 1, \ h_i < k_i\} \end{aligned}$$

Notice that  $\alpha + 1 \leq \beta \leq s + 1$ , and that if  $\beta = s + 1$ , then  $h_\beta = 0$ .

Then, we can consider  $h'$  defined by

$$\begin{aligned} h'_\alpha &= h_1 - 1, \quad h'_\beta = h_\beta + 1 \\ h'_i &= h_i, \quad \text{for any } i \neq \alpha, \beta \end{aligned}$$

The following lemma prove that

$$\text{Inv}_d(f; h) \cup \text{Inv}_d(f; h')$$

is connected. If  $h' = h_*$ , the proof is finished. If not, we consider  $h''$  in an analogous way, and so on.

**Lemma 4.6** *Let  $h$  be a partition of  $d$  compatible with  $k$ , such that, for some  $1 \leq \alpha < \beta \leq r$*

$$h_\alpha > h_\beta + 1, \quad h_\beta < k_\beta$$

*There exist an  $f$ -invariant subspace  $\mathcal{S} \subset \mathcal{Y}$  such that*

$$\mathcal{S} \in \text{Inv}_d(f; h) \cap \overline{\text{Inv}_d(f; h')}$$

*(where uperbar means “closure”).*

*Proof.* Let be  $w_i, f(w_i), \dots, f^{k_i-1}(w_i); \ 1 \leq i \leq r$ ,  $B$ -chains of  $f$ , as in section 2. Let consider  $v_i = f^{k_i-h_i}(w_i), \ 1 \leq i \leq s$  and the subspace  $\mathcal{S} \subset \mathcal{Y}$  spanned by the chains

$$v_i, f(v_i), \dots, f^{h_i-1}(v_i); \quad 1 \leq i \leq s$$

Clearly  $\mathcal{S} \in \text{Inv}_d(f; h)$ .

Now, let consider firstly the case  $h_\beta \neq 0$ . Then, let  $v' = f^{k_\beta-h_\beta-1}(w_\beta)$ , so that  $f(v') = v_\beta$ ; and, for each  $\varepsilon > 0$ , let  $\mathcal{S}_\varepsilon \subset \mathcal{Y}$  be the subspace spanned by the vectors

$$\begin{aligned} &v_i, f(v_i), \dots, f^{h_i-1}(v_i); \quad 1 \leq i \leq r, \quad i \neq \alpha, \beta \\ &f(v_\alpha), \dots, f^{h_\alpha-1}(v_\alpha) \\ &v_\beta, \dots, f^{h_\beta-1}(v_\beta) \\ &v_\alpha + \varepsilon v', \end{aligned}$$

$$v_\alpha + \varepsilon v', \quad f(v_\alpha) + \varepsilon v_\beta, \dots, f^{h_\beta}(v_\alpha) + \varepsilon f^{h_\beta-1}(v_\beta)$$

having length  $h_\beta + 1$ , because of  $f^{h_\beta+1}(v_\alpha) \in \mathcal{Y}$  (recall that  $h_\beta + 1 < h_\alpha$ , by hypothesis). Therefore, for any  $\varepsilon > 0$ , we have  $\mathcal{S}_\varepsilon \in \text{Inv}_d(f; h')$ . And obviously,  $\lim_{\varepsilon \rightarrow 0} \mathcal{S}_\varepsilon = \mathcal{S}$ .

Finally, if  $h_\beta = 0$  the proof is analogous by considering  $v' = f^{k_\beta-1}(w_\beta)$  and  $\mathcal{S}_\varepsilon \subset \mathcal{Y}$  the subspace spanned by the vectors

$$\begin{aligned} &v_i, f(v_i), \dots, f^{h_i-1}(v_i); \quad 1 \leq i \leq r, \quad i \neq \alpha, \beta \\ &f(v_\alpha), \dots, f^{h_\alpha-1}(v_\alpha) \\ &v_\alpha + \varepsilon v' \end{aligned}$$

■

**Remark 4.7** Notice that in the above proof, the hypothesis of  $f$  being observable has been used only to ensure that any restriction  $\hat{f} : \mathcal{S} \rightarrow \mathfrak{X}$  to an  $f$ -invariant subspace is also observable. Therefore, the proof can be easily generalized to prove that the union

$$\bigcup_{M, F} \text{Inv}_d(f; (M, F))$$

when  $(M, F)$  runs over all the pairs of matrices such that  $\begin{pmatrix} M \\ F \\ 0 \end{pmatrix}$  is an observable  $B$ -matrix having  $d$  columns. We will need this fact in the next theorem.

Finally, we have:

**Theorem 4.8** Let  $f : \mathcal{Y} \rightarrow \mathfrak{X}$  be such that  $f_\infty$  has only one eigenvalue. Then,  $\text{Inv}_d(f)$  is connected.

*Proof.* We recall that

$$\text{Inv}_d(f) = \bigcup_{M, F} \text{Inv}_d(f; (M, F))$$

where  $(M, F)$  runs over the set of pairs of matrices such that  $\begin{pmatrix} M \\ F \\ 0 \end{pmatrix}$  is a  $B$ -matrix having  $d$  columns. We will denote this set by  $B(d)$ . Then,  $M = \text{diag}\{M_0, M_\infty\}$  and  $F = \text{diag}\{F_0, 0\}$ , where  $\begin{pmatrix} M_0 \\ F_0 \\ 0 \end{pmatrix}$  is an observable  $B$ -matrix having  $h_0$  columns, and  $M_\infty$  is a Jordan  $h_\infty$ -square matrix with only one eigenvalue. If we denote these sets by  $OB(h_0)$  and  $\mathcal{J}(h_\infty)$  respectively we can identify

$$B(d) = \bigcup_{\substack{h_0, h_\infty \\ h_0 + h_\infty = d}} (OB(h_0) \times \mathcal{J}(h_\infty))$$

where

$$\text{Inv}_{h_0, h_\infty}(f) \equiv \bigcup_{OB(h_0) \times \mathcal{J}(h_\infty)} \text{Inv}_d(f; (M, F))$$

Firstly, we will see that each one of these sets is connected, and afterwards that it is so the union of them.

For each  $h_0, h_\infty$  fixed, with  $h_0 + h_\infty = d$ , we have

$$\begin{aligned} \text{Inv}_{h_0, h_\infty}(f) &= \bigcup_{OB(h_0), \mathcal{J}(h_\infty)} \theta(\text{Inv}_{h_0}(f; (M_0, F_0)) \times \text{Inv}_{h_\infty}(f; M_\infty)) = \\ &= \theta \left( \left[ \bigcup_{OB(h_0)} \text{Inv}_{h_0}(f; (M_0, F_0)) \right] \times \left[ \bigcup_{\mathcal{J}(h_\infty)} \text{Inv}_{h_\infty}(f; M_\infty) \right] \right) \end{aligned}$$

where  $\theta$  is the map in (4.1). To see that this set is connected it is sufficient to see that both sets in the claudators are so. The first one is connected by (4.7). And the second one because of, from (4.2), we have

$$\bigcup_{\mathcal{J}(h_\infty)} \text{Inv}_{h_\infty}(f; M_\infty) = \bigcup_{\mathcal{J}(h_\infty)} \text{Inv}_{h_\infty}(f_\infty, M_\infty) = \text{Inv}_{h_\infty}(f_\infty)$$

and the later is connected (see [8]). Finally, to see that the union

$$\bigcup_{h_0 + h_\infty = d} \text{Inv}_{h_0, h_\infty}(f)$$

is connected, it is sufficient to prove that, for any  $h_0, h_\infty$  ( $0 \leq h_0 < k_0, 0 < h_\infty \leq k_\infty$ ) there is an  $f$ -invariant subspace  $\mathcal{S}$  such that

$$\mathcal{S} \in \text{Inv}_{h_0, h_\infty}(f) \cap \overline{\text{Inv}_{h_0+1, h_\infty-1}(f)}$$

(where the uperbar means “closure”). In order to that, let be  $k_1 \geq \dots \geq k_r$  the  $B$ -indices of  $f$ ,  $\lambda$  the unic eigenvalue of  $f_\infty$ , and  $\delta_1 \geq \dots \geq \delta_\rho$  its Segre characteristic. For any  $f$ -invariant subspace  $\mathcal{S}$ , the  $B$ -matrix of the restriction  $\hat{f} : \mathcal{S} \rightarrow \mathcal{X}$  is determined by its  $B$ -indices  $h_1 \geq \dots \geq h_r$  and its Segre characteristic  $\eta_1 \geq \dots \geq \eta_\rho$ . Hence, we can note  $\text{Inv}_d(f; (M, F)) \equiv \text{Inv}_d(f; (h, \eta))$ , so that

$$\text{Inv}_{h_0, h_\infty}(f) = \bigcup_{h, \eta} \text{Inv}(f; (h, \eta))$$

where  $h$  runs over the partitions of  $h_0$  compatible with  $k$ , and  $\eta$  over the partitions of  $h_\infty$  compatible with  $\delta$ . Then, the following lemma ends the proof.  $\blacksquare$

**Lemma 4.9** *Given  $h_0, h_\infty$  with  $0 \leq h_0 < k_0, 0 < h_\infty \leq k_\infty$ , let be  $h$  a partition of  $h_0$  compatible with  $k$  and  $\alpha$  such that  $h_\alpha < k_\alpha$ , and let be  $\eta$  a partition of  $h_\infty$  compatible with  $\delta$  and  $\beta$  such that  $\eta_\beta > 0$ . Then, let consider  $h', \eta'$  defined by*

$$\begin{aligned} h'_\alpha &= h_\alpha + 1, & h'_i &= h_i & \text{if } i &\neq \alpha \\ \eta'_\beta &= \eta_\beta - 1, & \eta'_j &= \eta_j & \text{if } j &\neq \beta \end{aligned}$$

$$\mathcal{S} \in \text{Inv}(f; (h, \eta)) \cap \overline{\text{Inv}(f; (h', \eta'))}$$

(Hence:  $\mathcal{S} \in \text{Inv}_{h_0, h_\infty}(f) \cap \overline{\text{Inv}_{h_0+1, h_\infty-1}(f)}$ ).

*Proof.* Let be  $w_i, f(w_i), \dots, f^{k_i-1}(w_i)$ ,  $1 \leq i \leq r$ ,  $B$ -chains of  $f$ , and  $w_j, (f - \lambda \text{Id})(w_j), \dots, (f - \lambda \text{Id})^{\delta_j-1}(w_j)$ ,  $1 \leq j \leq \rho$ , a Jordan basis of  $f_\infty$ . Let consider

$$\begin{aligned} v_i &= f^{k_i-h_i}(w_i) & 1 \leq i \leq s \\ \nu_j &= (f - \lambda \text{Id})^{\delta_j-\eta_j}(w_j) & 1 \leq j \leq \sigma \end{aligned}$$

where we assume that  $h_s > h_{s+1} = 0$ ,  $\eta_\sigma > \eta_{\sigma+1} = 0$ . Let  $\mathcal{S} \subset \mathcal{Y}$  the subspace spanned by

$$\begin{aligned} v_i, f(v_i), \dots, f^{h_i-1}(v_i) & & 1 \leq i \leq s \\ \nu_j, (f - \lambda \text{Id})(\nu_j), \dots, (f - \lambda \text{Id})^{\eta_j-1}(\nu_j) & & 1 \leq j \leq \sigma \end{aligned}$$

Clearly  $\mathcal{S} \in \text{Inv}(f; (h, \eta))$ .

Now, let  $u' = f^{k_\alpha-h_\alpha-1}(w_\alpha)$ , so that  $f(u') = v_\alpha$ . Finally, for any  $\varepsilon > 0$ , let  $\mathcal{S}_\varepsilon$  be the subspace spanned by

$$\begin{aligned} &v_i, f(v_i), \dots, f^{h_i-1}(v_i), & 1 \leq i \leq s \\ &\nu_j, (f - \lambda \text{Id})(\nu_j), \dots, (f - \lambda \text{Id})^{\eta_j-1}(\nu_j), & 1 \leq j \leq \sigma, \quad j \neq \beta \\ &\varepsilon u' + \nu_\beta, (f - \lambda \text{Id})\nu_\beta, \dots, (f - \lambda \text{Id})^{\eta_\beta-1}(\nu_\beta) \end{aligned}$$

Obviously  $\mathcal{S} = \lim_{\varepsilon \rightarrow 0} \mathcal{S}_\varepsilon$ . And for all  $\varepsilon > 0$ ,  $\mathcal{S}_\varepsilon \in \text{Inv}(f; (h', \eta'))$ , because of  $\varepsilon u' + \nu_\beta$  spans a  $B$ -chain having length  $h_\alpha + 1$ :

$$\varepsilon u' + \nu_\beta, \varepsilon v_\alpha + f(\nu_\beta), \dots, \varepsilon f^{h_\alpha-1}(v_\alpha) + f^{h_\alpha}(\nu_\beta) \quad \blacksquare$$

## 5 Duality

Finally, we are going to see that the above results can be transferred by duality to the case of  $(A, B)$ -invariant subspaces, where  $(A, B) \in \text{Hom}(\mathbf{C}^{n+m}, \mathbf{C}^n)$ .

From a geometric point of view, we consider a  $(n+m)$ -dimensional vector space  $\mathcal{Z}$ , an  $m$ -dimensional subspace  $\mathcal{W} \subset \mathcal{Z}$ , and a linear map defined module it  $g : \mathcal{Z} \rightarrow \mathcal{Z}/\mathcal{W}$ . Its dual map  $g^t : \left(\frac{\mathcal{Z}}{\mathcal{W}}\right)^* \rightarrow \mathcal{Z}^*$  falls on the case studied above by means of the identification:  $\left(\frac{\mathcal{Z}}{\mathcal{W}}\right)^* \simeq \widetilde{\mathcal{W}} = \{\omega \in \mathcal{Z}^* : \text{Ker } \omega \supset \mathcal{W}\}$ .

In these terms, a subspace  $\mathcal{S}$ , with  $\mathcal{W} \subset \mathcal{S} \subset \mathcal{Z}$  is called  $g$ -invariant if  $g(\mathcal{S}) \subset \frac{\mathcal{S}}{\mathcal{W}} + g(\mathcal{W})$ . In fact, this definition corresponds to the matricial one in [7] because of the following lemma.

**Lemma 5.1** *With the above notation,  $\mathcal{S}$  is a  $g$ -invariant subspace if and only if  $\tilde{\mathcal{S}} \subset \tilde{\mathcal{W}}$  is a  $g^t$ -invariant subspace.*

$$g^t(\tilde{\mathcal{S}}) \cap \tilde{\mathcal{W}} \subset \tilde{\mathcal{S}}.$$

Then:  $\mathcal{S} \subset g^t(\tilde{\mathcal{S}}) + \mathcal{W}$ . Hence:

$$g(\mathcal{S}) \subset g\left(g^t(\tilde{\mathcal{S}})\right) + g(\mathcal{W}) \subset \frac{\mathcal{S}}{\mathcal{W}} + g(\mathcal{W}).$$

Conversely, if  $\mathcal{S}$  is a  $g$ -invariant subspace, and  $\omega \in g^t(\tilde{\mathcal{S}}) \cap \tilde{\mathcal{W}}$ , it is straightforward that  $\text{Ker } \omega \supset \mathcal{S}$ . ■

That leads to consider  $Gr_{n+m-d}(\mathcal{Z}; \mathcal{W})$  the grassman manifold of  $(n + m - d)$ -dimensional subspaces  $\mathcal{S}$  of  $\mathcal{Z}$  such that  $\mathcal{S} \supset \mathcal{W}$ , which is naturally diffeomorphic to  $Gr_{n-d}(\mathcal{Z}/\mathcal{W})$ . Because of the above lemma, we have:

**Proposition 5.2** *Let be the diffeomorphism*

$$\begin{aligned} \varphi : Gr_{n+m-d}(\mathcal{Z}; \mathcal{W}) &\longrightarrow Gr_d(\mathcal{Z}^*) \\ \varphi(\mathcal{S}) = \tilde{\mathcal{S}} &= \{\omega \in \mathcal{Z}^* : \text{Ker } \omega \supset \mathcal{S}\} \end{aligned}$$

*Then:*

$$\varphi(\text{Inv}_{n+m-d}(g)) = \text{Inv}_d(g^t)$$

Through this diffeomorphism  $\varphi$  we can stratify in a natural way  $\text{Inv}_{n+m-d}(g)$  and then, the results in sections 2, 3 and 4 concerning  $\binom{A}{C}$ -invariants subspaces can be transferred to the case of  $(A, B)$ -invariant subspaces.

**Remark 5.3** A nice connection between the set of  $\binom{A}{C}$ -invariant subspaces of codimension  $k$  for a generic pair  $\binom{A}{C}$  and the orbit space of  $k$ -dimensional observable pairs is shown in [7]. It would be very interesting, as the authors of this paper point out, to go further in the investigation of this link for general pairs  $\binom{A}{C}$ .

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